Sesión Especial 5

Recent developments on Convexity

Organizadores:

- Mónica Blanco (Universidad de Cantabria)
- Bernardo González Merino (Universidad de Murcia)
- Rafael Villa (Universidad de Sevilla)

Summary: Convex Geometry is an increasingly growing area of mathematics. Even though its roots go back to the ancient greeks, the core of the theory (the Brunn-Minkowski theory) still attracts many new researchers. Convexity notions and techniques can be found in asymptotic geometric analysis, affine invariant inequalities, discrete geometry, geometry of numbers, geometry of Banach spaces, Optimization, or even linear and convex programming.

The aim of this special session is to gather experts from different research groups, both from Spain and other countries. We expect those experts to deliver research talks, based on their novel mathematical results. We hope that the encounter will boost possible new collaborations and connections among some of the speakers and the audience. Indeed, several speakers are either PhD students or belong to the early stages of their postdoctoral lifes.

Programa

Lunes, 22 de enero:

16:00 – 16:30	César Rosales (Universidad de Granada) *The anisotropic isoperimetric problem for convex cones*
16:30 – 17:00	Jesús Yepes (Universidad de Murcia) *On complemented Brunn-Minkowski type inequalities*
17:00 – 17:30	Eduardo Lucas (Universidad Politécnica de Cartagena) *Compressions: An application to discrete isoperimetric inequalities*
17:30 – 18:00	Ignacio Villanueva (Universidad Complutense de Madrid) *Measure valued valuations on star bodies*

Martes, 23 de enero:

11:30 – 12:00	Eugenia Saorín-Gómez (Universität Bremen) *On refinements of classical inequalities under projection assumptions*
12:00 – 12:30	Julián Haddad (Universidad de Sevilla) *Fiber symmetrization and the Rogers-Brascamp-Lieb-Luttinger inequality*
12:30 – 13:00	Luis Crespo (Universidad de Cantabria) *Integer points in smooth polytopes and implications in symplectic geometry*
13:00 – 13:30	Antonio Cañete (Universidad de Sevilla) *The Cheeger set for a rotationally symmetric planar convex body*
16:00 – 16:30	Gil Solanes (Universidad Autónoma de Barcelona) *Valoraciones en variedades de Kähler*
16:30 – 17:00	Javier Martín Goñi (Universidad de Zaragoza) *High-dimensional limit theorems for the distance between random vectors in l_p^n-balls*
17:00 – 17:30	Florian Grundbacher (University of Munich) *p-Means of Convex Bodies and a new Suggestion for the Geometric Mean of Convex Bodies*
17:30 – 18:00	David Alonso-Gutiérrez (Universidad de Zaragoza) *A discrete approach to Zhang's inequality*

The anisotropic isoperimetric problem for convex cones

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Abstract: Given a norm Ψ in $\mathbb{R}^{n+1}$ and a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the *anisotropic area* of Σ is defined as $A_\Psi(\Sigma) = \int_\Sigma \Psi(N)\, d\Sigma$, where N is a unit normal over Σ and $d\Sigma$ denotes the Euclidean area element. The *anisotropic isoperimetric problem* in $\mathbb{R}^{n+1}$ studies sets minimizing the anisotropic area of the boundary while enclosing a fixed Euclidean volume. It is well-known that the solutions to this problem coincide, up to translations and dilations centered at the origin, with the centrally symmetric convex body K supported by Ψ.
In this talk we will analyze the same problem within a *convex open cone* $\mathcal{C} \subset \mathbb{R}^{n+1}$. So, we seek compact hypersurfaces $\Sigma \subset \overline{\mathcal{C}}$ with $\partial\Sigma = \Sigma \cap \partial\mathcal{C}$ and minimizing $A_K(\Sigma)$ while separating a fixed volume in $\mathcal{C}$. We will review known results about the existence and characterization of solutions in this setting. Our main contribution shows that, under additional regularity conditions, any second order minimum for this problem is contained in ∂K (up to a translation and a dilation about the origin).

On complemented Brunn-Minkowski type inequalities

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Abstract: A measure μ on $\mathbb{R}^n$ is said to be q-concave if it satisfies a q-Brunn-Minkowski inequality, namely,

$$\mu\big((1-\lambda)A+\lambda B\big)\geq\big((1-\lambda)\mu(A)^q+\lambda\mu(B)^q\big)^{1/q}$$

for all measurable sets $A,B\subset\mathbb{R}^n$ with $\mu(A)\mu(B)>0$ such that $(1-\lambda)A+\lambda B$ is also measurable, and all $\lambda\in(0,1)$.

Following the duality of concave and convex functions, it is natural to wonder about a q-complemented Brunn-Minkowski inequality, i.e., whether

$$\mu\big(\mathbb{R}^n\setminus\big((1-\lambda)A+\lambda B\big)\big)\leq\big((1-\lambda)\mu\big(\mathbb{R}^n\setminus A\big)^q+\lambda\mu\big(\mathbb{R}^n\setminus B\big)^q\big)^{1/q},$$

provided that $\mu\big(\mathbb{R}^n\setminus A\big),\mu\big(\mathbb{R}^n\setminus B\big)<+\infty$.

When (μ is finite and) $q=1$ both conditions above are trivially equivalent, but this equivalence is no longer true in general for other values of q. However, Milman and Rotem in 2014 showed that under certain assumptions of concavity and homogeneity for the density of μ such an inequality holds. In particular, the restriction of the Lebesgue measure $\mathrm{vol}(\cdot)$ to a convex cone C (which is its support) satisfies the latter inequality for $q=1/n$ and any $A,B\subset C$ with $\mathrm{vol}\big(C\setminus A\big),\mathrm{vol}\big(C\setminus B\big)<+\infty$. This case was later studied also by Schneider in 2018, who gave a different proof and characterized its equality case when A and B are convex.

In this talk we will discuss about different functional and geometric forms of complemented Brunn-Minkowski type inequalities for certain absolutely continuous measures on $\mathbb{R}^n$ containing, among others, both the volume and the standard Gaussian measure γ_n. In particular, we will show the connection between this family of complemented Brunn-Minkowski inequalities and that of dual Brunn-Minkowski inequalities (involving the radial sum).

This is about joint work in progress with A. Zvavitch.

Acknowledgments: This work was partially developed during a research stay of the first author at Kent State University, supported by AEI "RYC2021-034858-I" project.

Compressions: An application to discrete isoperimetric inequalities

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Abstract: The *isoperimetric inequality* is perhaps one of the most classical results in geometry, and states that for any given positive volume (Lebesgue measure), Euclidean balls are the sets that minimize the surface area measure among the family of compact sets with said volume. Copious extensions and generalizations to diverse contexts and branches in mathematics have been obtained since its original inception in the Hellenistic period. In this talk, we will be specifically interested in versions of this inequality in the discrete setting, in particular, for subsets of lattices, such as $\mathbb{Z}^n$.

We will recall the notion of *compressed sets* in this setting, and the corresponding process of *compression*, which has been largely exploited in the context of discrete geometry and number theory, and in particular, in the study of discrete isoperimetric inequalities. We will present some recent discrete versions of this inequality, and a characterization of the equality case in special cases, both of which heavily depend on the process of compression.

This is joint work with David Iglesias (Universidad de Murcia).

Acknowledgments: We would like to express our gratitude to María A. Hernández Cifre and J. Yepes Nicolás, who brought this topic to our attention and greatly helped us polish the work.

Measure valued valuations on star bodies

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Abstract: We study valuations defined on the star bodies of $\mathbb{R}^n$ with values on the space of measures on the unit sphere S^{n-1}. In particular, we provide an integral representation extending earlier results on scalar valuations on star bodies. Particular attention is devoted to equivariant valuations. A functional characterization of dual area measures is also provided.

On refinements of classical inequalities under projection assumptions

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Abstract: Based on projection assumptions, we will discuss some classical and new "linearrefinements of inequalities within the Brunn-Minkowski and elliptic Brunn-Minkowski theory, with the aim of exploring parallelisms and differences of both theories.

Fiber symmetrization and the Rogers-Brascamp-Lieb-Luttinger inequality

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Abstract: We prove a Rogers-Brascamp-Lieb-Luttinger inequality for functions defined in the space of $n \times m$ matrices, using a particular form of fiber-symmetrization. Some applications on symmetrization of matrix norms are given. We also discuss a conjectured inequality by Schneider, on the higher-order difference body.

Integer points in smooth polytopes and implications in symplectic geometry

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Abstract: This talk presents several open problems concerning integer points of polytopes arising in symplectic toric geometry, and the solutions for some of them. Ewald's Conjecture from 1988 states that if P is a monotone n-polytope in $\mathbb{R}^n$ then the set $\mathbb{Z}^n \cap P \cap (-P)$ contains a unimodular basis of the lattice $\mathbb{Z}^n$. In 2009 Nill proposed a generalization of Ewald's Conjecture, which says that if P is an n-dimensional lattice smooth polytope in $\mathbb{R}^n$ then $\mathbb{Z}^n \cap P \cap -P$ contains a unimodular basis of $\mathbb{Z}^n$. I will present proofs joint with Álvaro Pelayo of the following cases of these conjectures: Ewald's conjecture for n-polytopes which do not recursively contain unimodular triangles and Nill's conjecture for $n = 2$. A full version of this talk is available at arXiv:2310.10366.

The Cheeger set for a rotationally symmetric planar convex body

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Abstract: Let Ω be a k-rotationally symmetric planar convex body, and let C_Ω be the *Cheeger set* associated to Ω (that is, among all the subsets of Ω, C_Ω is the one minimizing the quotient of the perimeter over the enclosed area).
In this talk, we will describe some properties of C_Ω. In particular, we will show that C_Ω is k-rotationally symmetric. Additionally, by introducing previously the notions of *dots* and *edges* of Ω, we will prove that the boundary of C_Ω touches all the edges of Ω (in other words, Ω is a Cheeger-regular set, in view of a definition from a paper by Kawohl and Lachand-Robert [2]).

Referencias

[1] A. Cañete (2022). Cheeger sets for rotationally symmetric planar convex bodies, Results Math., 77, paper n. 9, 15 pp.

[2] B. Kawohl, T. Lachand-Robert (2006). Characterization of Cheeger sets for convex subsets of the plane, Pacific J. Math, 225, 103–118.

Valoraciones en variedades de Kähler

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Abstract: En el marco de la teoría de valoraciones en variedades, los funcionales de Lipschitz-Killing generan un álgebra asociada canónicamente a cualquier variedad de Riemann. En el caso de la esfera y del espacio hiperbólico, esta álgebra coincide con el espacio de valoraciones invariantes por isometrías. Esto implica, por ejemplo, que las álgebras de valoraciones invariantes de la esfera y del espacio euclídeo son isomorfas. En la charla presentaremos un álgebra canónica de valoraciones asociada a cualquier variedad de Kähler [2]. En el proyectivo complejo, esta álgebra coincide con el espacio de valoraciones invariantes por isometrías, lo que da lugar a un isomorfismo canónico entre las álgebras de valoraciones invariantes de los espacios proyectivo y euclídeo complejos. A su vez, esto explica de forma satisfactoria algunos resultados un tanto sorprendentes en geometría integral hermítica [1].

Referencias

[1] A. Bernig, J.H.G. Fu, G. Solanes (2014). Integral geometry of complex space forms. Geom. Funct. Anal. 24, 402-492.

[2] A. Bernig, J.H.G. Fu, G. Solanes, T. Wannerer (2023). The Weyl tube theorem for Kähler manifolds arXiv:2209.05806

High-dimensional limit theorems for the distance between random vectors in l_p^n-balls

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Resumen:
Let $p \geq 1$, and let $g_1, ..., g_n$ be independent copies of a random variable g with density with respect to the lebesgue measure given by

$$\frac{e^{-|t|^p}}{2\Gamma(1+1/p)}.$$

Then, the random vector $G = (g_1, ..., g_n)$ satisfies that $\frac{G}{\|G\|_p}$ is uniformly distributed on the boundary of the l_p^n unit ball, denoted as ∂B_p^n. Let G' be an independent copy of G. Then,

$$\left\| \frac{G}{\|G\|_p} - \frac{G'}{\|G'\|_p} \right\|_2$$

is the distance between two uniformly distributed random vectors on ∂B_p^n.
In [1], Hammersley proved a central limit theorem for the case $p = 2$. In particular, he proved that as the dimension n tends to infinity, the distance between uniformly distributed random vectors on hyperspheres tends to $N\left(\frac{1}{\sqrt{2}}, \frac{1}{2n}\right)$, where $N(\mu, \sigma^2)$ is a Gaussian random variable with mean μ and variance σ^2.
In this talk we will study the cases where the random vectors are uniformly distributed on the boundary of B_p^n, for $p \geq 1$, and obtain their respective central limit theorems.

Referencias

[1] J. M. Hammersley (1950). The Distribution of Distance in a Hypersphere. The Annals of Mathematical Statistics, vol. 21, no. 3, pp. 447-52.

p-Means of Convex Bodies and a new Suggestion for the Geometric Mean of Convex Bodies

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Abstract: In light of the log-Brunn-Minkowski conjecture, various attempts have been made to define the geometric mean of convex bodies. Many of these constructions are fairly complex and/or fail to satisfy some natural properties one would expect of such a mean. To improve our understanding of potential geometric mean definitions, we study the closely related p-means of convex bodies, with the usual definition extended to two series ranging over all $p \in [-\infty, \infty]$. We characterize their equality cases and obtain (in almost all instances tight) inequalities that quantify how well these means approximate each other. Based on our findings, we propose a fairly simple definition of the geometric mean that satisfies the properties considered in recent literature, and discuss potential axiomatic characterizations. Finally, we conclude that some of these properties are incompatible with approaches to proof the log-Brunn-Minkowski conjecture via geometric means.

A discrete approach to Zhang's inequality

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Abstract: We will consider the inclusion relation between the n-th Ball body of the covariogram function g_K of a convex body $K \subseteq \mathbb{R}^n$ and its polar projection body, $\binom{2n}{n}^{\frac{1}{n}} K_n(g_K) \subseteq n(\mathrm{vol}_n(K))\Pi^*(K)$, which was proved by Gardner and Zhang and leads to Zhang's inequality. We will provide a slightly different proof of such inclusion which will only make use of Berwald's inequality in the positive range of the parameters involved. With this approach we will be able to provide discrete analogues, where we consider the lattice point enumerator measure, instead of the Lebesgue measure. Such discrete inequalities will allow us to recover the continuous ones.